Suggested solution to Assignment 2

1. (a)

$$\begin{array}{c|cccc|c} & & & & & \min \\ & 5 & 1 & -2 & 6 & -2 \\ & -1 & 0 & 1 & -2 & -2 \\ & 3 & 2 & 5 & 4 & 2 \\ \hline \max & 5 & 2 & 5 & 6 & \end{array}$$

Both the maximin and minimax are 2. Therefore the entry $a_{32} = 2$ is a saddle point. The value of the game is 2.

(b)

$$\begin{array}{c|cccc|c} & & & & & \min \\ & -4 & 5 & -3 & -3 & -4 \\ & 0 & 1 & 3 & -1 & -1 \\ & -3 & -1 & 2 & -5 & -5 \\ & 2 & -4 & 0 & -2 & -4 \\ \hline \max & 2 & 5 & 3 & -1 & \end{array}$$

Both the maximin and minimax are -1. Therefore the entry $a_{24} = -1$ is a saddle point. The value of the game is -1.

2. (a)

$$A = \begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix} \quad \begin{matrix} -6 \\ 4 \end{matrix} \times \begin{matrix} 4 \\ 6 \end{matrix} \quad \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$\begin{matrix} -1 & 9 \\ & \times & \\ 9 & 1 \end{matrix}$$

$$\left(\tfrac{9}{10}, \tfrac{1}{10}\right)$$

So the maximin strategy for the row player is $\left(\frac{2}{5}, \frac{3}{5}\right)$.
The minimax strategy for the column player is $\left(\frac{9}{10}, \frac{1}{10}\right)$.
The value of the game is $v = \left(\frac{2}{5}, \frac{3}{5}\right)\begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} \frac{9}{10} \\ \frac{1}{10} \end{pmatrix} = \frac{8}{5}$.

(b)

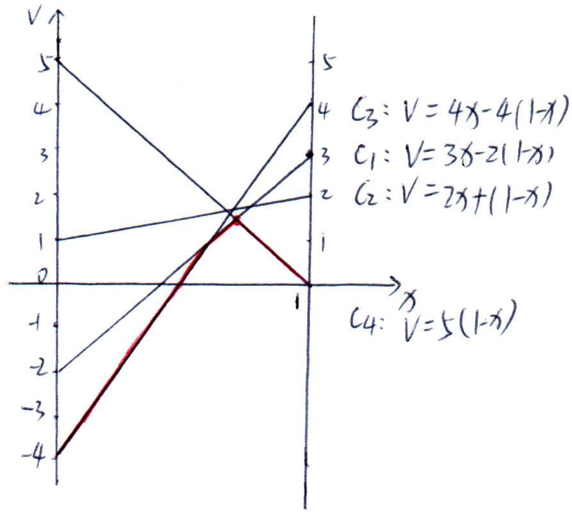
$$A = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} \quad \begin{matrix} 4 \\ -6 \end{matrix} \times \begin{matrix} 6 \\ 4 \end{matrix} \quad \begin{pmatrix} \frac{3}{5} \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{matrix} 5 & -5 \\ & \times & \\ 5 & 5 \end{matrix}$$

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

So the maximin strategy for the row player is $\left(\frac{3}{5}, \frac{2}{5}\right)$.
The minimax strategy for the column player is $\left(\frac{1}{2}, \frac{1}{2}\right)$.
The value of the game is $v = \left(\frac{3}{5}, \frac{2}{5}\right)\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = 1$

(c) $A=\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$



By drawing the lower envelope, the maximum point of the lower envelope is the intersection point of C_1 and C_4. By solving

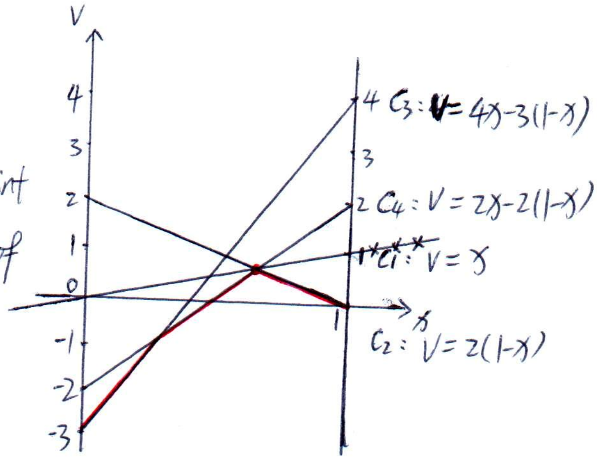
$$\begin{cases} C_1: v=3x-2(1-x) \\ C_4: v=5(1-x) \end{cases}$$

$x=0.7,\ v=1.5$

For the minimax strategy: $\begin{pmatrix} 3 & 0 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_4 \end{pmatrix}=\begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix} \Rightarrow y_1=y_4=0.5.$

Hence the maximin strategy for the row player is $(0.7, 0.3)$; the minimax strategy for the column player is $(0.5, 0, 0, 0.5)$; and the value is 1.5.

(d) $A=\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & -3 & -2 \end{pmatrix}$



By drawing the lower envelope, the maximum point of the lower envelope is the intersection point of C_1, C_2 and C_4. By solving

$$\begin{cases} C_1: v=x \\ C_2: v=2(1-x) \\ C_4: v=2x-2(1-x) \end{cases}$$

$x=\frac{2}{3},\ v=\frac{2}{3}$

For the minimax strategy: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_4 \end{pmatrix}=\begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

Note that we have added the equation $y_1+y_2+y_4=1$ to exclude the solutions which are not probability vectors. Using row operation, we obtain the row echelon for

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & \frac{2}{3} \\ 0 & 2 & -2 & \frac{2}{3} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & \frac{2}{3} \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The non-negative solution to the system of equations is $(y_1, y_2, y_4)=(\frac{2}{3}-2t, \frac{1}{3}+t, t)$, $0\le t\le \frac{1}{3}$.

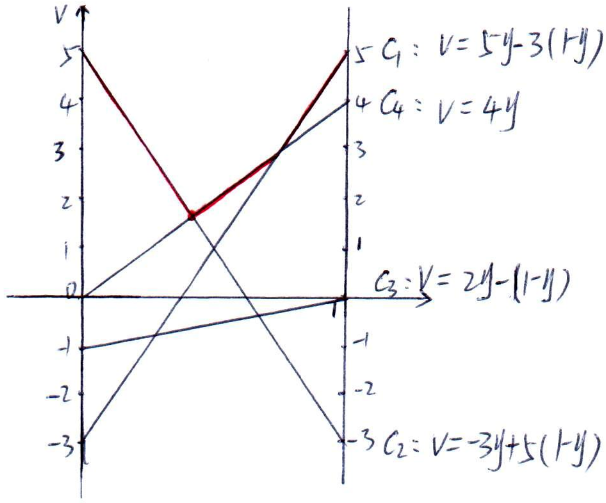
Hence the column player has minimax strategies $q=(\frac{2}{3}-2t, \frac{1}{3}+t, 0, t)$ for $0\le t\le \frac{1}{3}$.

In particular, $(\frac{2}{3}, \frac{1}{3}, 0, 0)$ and $(0, \frac{2}{3}, 0, \frac{1}{3})$ are minimax strategies for the column player;

The maximin strategy for the row player is $(\frac{2}{3}, \frac{1}{3})$;

The value of the game is $\frac{2}{3}$.

(e) $A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 2 & -1 \\ 4 & 0 \end{pmatrix}$



The upper envelope is shown in the right figure.

Solving $\begin{cases} C_2: v = -3y + 5(1-y) \\ C_4: v = 4y \end{cases}$

$y = \frac{5}{12}$, $v = \frac{5}{3}$

$\begin{pmatrix} -3 & 5 \\ 4 & 0 \end{pmatrix}$ -8 4 4 8 $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$

-7 5

5 7

$\left(\frac{5}{12}, \frac{7}{12}\right)$

Therefore the maximin strategy for the row player and the minimax strategy for the column player are $\left(0, \frac{1}{3}, 0, \frac{2}{3}\right)$ and $\left(\frac{5}{12}, \frac{7}{12}\right)$ respectively; the value is $\frac{5}{3}$.

(f) $A = \begin{pmatrix} 5 & -2 & 3 \\ 3 & -1 & 4 \\ 0 & 3 & 1 \end{pmatrix}$ A is nonsingular

Using row operation, we obtain

$$\left(\begin{array}{ccc|ccc} 5 & -2 & 3 & 1 & 0 & 0 \\ 3 & -1 & 4 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{32} & -\frac{11}{32} & \frac{5}{32} \\ 0 & 1 & 0 & \frac{3}{32} & -\frac{5}{32} & \frac{11}{32} \\ 0 & 0 & 1 & -\frac{9}{32} & \frac{15}{32} & -\frac{1}{32} \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} \frac{13}{32} & -\frac{11}{32} & \frac{5}{32} \\ \frac{3}{32} & -\frac{5}{32} & \frac{11}{32} \\ -\frac{9}{32} & \frac{15}{32} & -\frac{1}{32} \end{pmatrix}$, $\mathbf{1}^T A^{-1} \mathbf{1} = \frac{21}{32}$ $\quad (\mathbf{1} = (1, 1, 1)^T)$

By tutorial notes, $v = 1/\mathbf{1}^T A^{-1} \mathbf{1} = \frac{32}{21} \neq 0$, $p^T = \mathbf{1}^T A^{-1} = \left(\frac{7}{21}, -\frac{1}{21}, \frac{15}{21}\right)^T$.

Since p has negative component, there is a theorem showing that an arbitrary $m \times n$ matrix game whose value is not zero may be solved by choosing some suitable square submatrix and checking the resulting optimal strategies for the whole matrix.

By checking, we choose the submatrix $A' = \begin{pmatrix} 5 & -2 \\ 0 & 3 \end{pmatrix}$ 7 -3 3 7 $\begin{pmatrix} \frac{3}{10} \\ \frac{7}{10} \end{pmatrix}$

5 -5

5 5

$\left(\frac{1}{2}, \frac{1}{2}\right)$

Hence a maximin strategy for the row player is $p=(\frac{3}{10},0,\frac{7}{10})$; a minimax strategy for the column player is $q=(\frac{1}{2},\frac{1}{2},0)$; the value is $v=\frac{5\times3-0}{5+3-(-2)}=\frac{3}{2}$.

One may check the result by the following calculations.

$$pA=(\tfrac{3}{10},0,\tfrac{7}{10})\begin{pmatrix}5&-2&3\\3&-1&4\\0&3&1\end{pmatrix}=(\tfrac{3}{2},\tfrac{3}{2},\tfrac{8}{5})$$

$$Aq^T=\begin{pmatrix}5&-2&3\\3&-1&4\\0&3&1\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\1\\\frac{3}{2}\end{pmatrix}$$

One sees that the row player may guarantee that his payoff is at least $\frac{3}{2}$ by using $p=(\frac{3}{10},0,\frac{7}{10})$ and the column player may guarantee the payoff to the row player is at most $\frac{3}{2}$ by using $q=(\frac{1}{2},\frac{1}{2},0)$.

(g)

$$A=\begin{array}{c|cccc|c} & & & & & \min\\ \hline & 5 & 1 & -2 & 6 & -2\\ & -1 & 0 & 1 & -2 & 2\\ & 3 & 2 & 5 & 4 & 2\\ \hline \max & 5 & 2 & 5 & 6 & \end{array}$$

Both the maximin and minimax are 2. Hence $a_{32}=2$ is a saddle point.

Then the value of the game is 2.

Obviously, the maximin strategy for the row player is $(0,0,1)$;

the minimax strategy for the column player is $(0,1,0,0)$.

3. The game matrix is $A=\begin{pmatrix}-3&2\\9&-8\end{pmatrix}$.

$$\begin{array}{cc} \begin{pmatrix}-3&2\\9&-8\end{pmatrix} & \begin{array}{ccc}-5 & & 17\\ 17 & & 5\end{array} \quad \begin{pmatrix}\frac{17}{22}\\\frac{5}{22}\end{pmatrix}\\ \begin{array}{cc}-12 & 10\\ 10 & 12\end{array} & \\ (\frac{5}{11},\frac{6}{11}) & \end{array}$$

Therefore the value of the game is $(\frac{17}{22},\frac{5}{22})\begin{pmatrix}-3&2\\9&-8\end{pmatrix}\begin{pmatrix}\frac{5}{11}\\\frac{6}{11}\end{pmatrix}=-\frac{3}{11}$;

the optimal strategy for Raymond is $(\frac{17}{22},\frac{5}{22})$;

the optimal strategy for Calvin is $(\frac{5}{11},\frac{6}{11})$.

4\. (a)
$$\begin{array}{c|cc} & \text{Becky 1} & \text{Becky 2} \\ \hline \text{Alex 1} & 3 & -1 \\ \text{Alex 2} & -1 & 11 \end{array}$$
, i.e. the game matrix is $\begin{pmatrix} 3 & -1 \\ -1 & 11 \end{pmatrix}$.

$$\begin{pmatrix} 3 & -1 \\ -1 & 11 \end{pmatrix} \quad \begin{matrix} 4 \\ -12 \end{matrix} \times \begin{matrix} 12 \\ 4 \end{matrix} \quad \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$\begin{matrix} 4 & & -12 \\ & \times & \\ 12 & & 4 \end{matrix}$$

$$\left(\tfrac{3}{4}, \tfrac{1}{4}\right)$$

Hence the optimal strategies for Alex and Becky are both $\left(\frac{3}{4}, \frac{1}{4}\right)$.

(b) By (a), the value of the game is $v = \left(\frac{3}{4}, \frac{1}{4}\right)\begin{pmatrix} 3 & -1 \\ -1 & 11 \end{pmatrix}\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} = 2$.

Hence to make the game fair, $k = v = 2$.

5\. The game matrix is
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

We may delete the first row and the last row since they are dominated by the third and the fifth row respectively to get the reduced matrix
$$A' = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Then we may delete the third, and the fifth columns since they are dominated by the first, the second and the fourth the last columns respectively to get the reduced matrix
$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally we may delete the third row since it is dominated by any other row.

Hence the matrix A is reduced to the 4×4 **diagonal** matrix
$$A''' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$, where diagonal entries $d_i = 1$, $i = 1, 2, 3, 4$.

Using the principle of indifference, $V = 1/\mathbf{1}^T A'''^{-1}\mathbf{1} = \left(\sum_{i=1}^{4} 1/d_i\right)^{-1} = 1$. $(\mathbf{1} = (1,1,1,1)^T)$

And $p = V A'''^{-T}\mathbf{1} = V(1/d_1, \cdots 1/d_4) = (1, 1, 1, 1)$

Similarly, $q = V A'''^{-1}\mathbf{1} = V(1/d_1, \cdots 1/d_4) = (1, 1, 1, 1)$.

Therefore, the value of A is 1; the optimal strategies for player I and player II are (0, 1, 0, 1, 1, 1, 0) and (1, 1, 0, 0, 0, 1, 1) respectively.

6. Assume that II has optimal strategy giving positive weight in each entry. By principle of indifference, I's optimal strategy p satisfies

$$\sum_{i=1}^{m} p_i a_{ij} = V, \quad j = 1, 2, \cdots, m.$$

Thus $p_1 = V$, $-2p_1 + p_2 = V$, $3p_1 - 2p_2 + p_3 = V$, $-4p_1 + 3p_2 - 2p_3 + p_4 = V$.

Solving $p_1 = V$, $p_2 = 3V$, $p_3 = 4V$, $p_4 = 4V$

Since $\sum_{i=1}^{4} p_i = 1$, we get $12V = 1$, thus $V = 1/12$.

And $p = (p_1, p_2, p_3, p_4) = (1/12, 1/4, 1/3, 1/3)$.

Similar argument shows that $q = (1/3, 1/3, 1/4, 1/12)$.

Since both p and q are nonnegative, both are optimal strategies and $1/12$ is the value of the game.

7. By the condition, the game matrix is

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & -1 & 2 & 2 & 2 \\ 1 & 0 & -1 & 2 & 2 \\ -2 & 1 & 0 & -1 & 2 \\ -2 & -2 & 1 & 0 & -1 \\ -2 & -2 & -2 & 1 & 0 \end{pmatrix} \end{array}$$

This game is symmetric, so the value is zero.

Note that row 1 dominates rows 4 and 5, and column 1 dominated columns 4 and 5.

We only need to consider the upper left 3×3 submatrix $\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}$.

Assume that II has optimal strategy q' so that $q_1' > 0$, $q_2' > 0$, $q_3' > 0$.

By the principle of indifference, we have

$$p_2' - 2p_3' = 0, \quad -p_1' + p_3' = 0, \quad 2p_1' - p_2' = 0$$

Together with the condition $p_1' + p_2' + p_3' = 0$, we have $p_1' = 1/4$, $p_2' = 1/2$, $p_3' = 1/4$.

Similarly, we can get $q_1' = 1/4$, $q_2' = 1/2$, $q_3' = 1/4$.

Hence, the optimal strategies are $p = q = (1/4, 1/2, 1/4, 0, 0)$.

8. (a)

$$\begin{array}{ccc} & & \text{min} \\ \begin{pmatrix} -3 & 1 \\ c & -2 \end{pmatrix} & & \begin{array}{c} -3 \\ x \end{array} \\ \text{max } \; y \quad 1 & & \end{array}$$

(i) If $c \le -3$, then $x = c$, so minimax is -3; and $y = -3$, so the maximin is -3.

Hence we see that minimax = maximin, i.e. A has a saddle point if $c \le -3$

(ii) If $-3 < c \le -2$, then $x = c$, so minimax is c; and $y = c$, so maximin is c.

Hence A has a saddle point if $-3 < c \le -2$.

(iii) If $c > -2$, then $x = -2$, so minimax is -2; and $y = c$, so maximin > -2

Hence A has no saddle point if $c > -2$.

Therefore, if $c \le -2$, then A has a saddle point.

(b) (i) By the hypothesis, the value of A is 0.

Thus $v = \dfrac{(-3)(-2) - 1 \cdot c}{-3-2-1-c} = 0 \implies c = 6$.

(ii) By (i), $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}$.

Hence the maximin strategy for the row player is $p = \left(\dfrac{-2-6}{-3-2-6-1}, \dfrac{-3-1}{-3-2-6-1} \right) = \left(\dfrac{2}{3}, \dfrac{1}{3}\right)$;

the minimax strategy for the column player is $q = \left(\dfrac{-2-1}{-3-2-6-1}, \dfrac{-3-6}{-3-2-6-1} \right) = \left(\dfrac{1}{4}, \dfrac{3}{4}\right)$.

9. Method 1:

Since $A^T = -A \implies A = -A^T$, the value of A and $-A^T$ are the same.

Hence $v = -v$, i.e, $2v = 0 \implies v = 0$

Therefore, the value of A is zero.

Method 2:

Let p be an optimal strategy for I. If II uses the same strategy, then

$p^T A p = \sum \sum p_i a_{ij} p_j = \sum \sum p_i (-a_{ji}) p_j = -\sum \sum p_j a_{ji} p_i = -p^T A p$.

Hence $p^T A p = 0$. This shows that the value $v \le p^T A p = 0$, i.e. $v \le 0$.

A symmetric argument shows that $v \ge q A q^T = 0$, i.e. $v \ge 0$.

Therefore, the value of A is $v = 0$.

10: (a) Set

$$
\begin{array}{cc}
D = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_k & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-1} & \\ & & & & & & \lambda_n \end{pmatrix} & \begin{array}{c} \min \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_{n-1} \\ x_n \end{array} \\
\max \; y_1 \; y_2 \; \cdots \; y_k \; \cdots \; y_{n-1} \; y_n &
\end{array}
$$

Since $\lambda_1 \le 0$, $\lambda_n > 0$, we get $x_1 = \lambda_1 \le 0$, $x_n = 0$, and $x_k = \min\{0, \lambda_k\}$, $k = 2, 3, \cdots, n-1$

For $k = 2, 3, \cdots n-1$, $x_k \le 0$, hence the minimax is $x_n = 0$.

Similarly, we can get $y_1 = 0$, $y_n = \lambda_n > 0$ and $y_k = \max\{0, \lambda_k\}$, $k = 2, 3, \cdots n-1$.

For $k = 2, 3, \cdots n-1$, $y_k \ge 0$, hence the maximin is $y_1 = 0$

Therefore the entry $a_{n1} = 0$ is a saddle point of D, then the value of the zerosum game with game matrix D is 0.

(b) Since $\lambda_1 > 0$, and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, we have $\lambda_n \ge \lambda_{n-1} \ge \cdots \ge \lambda_1 > 0$.

Assume II has optimal strategy with positive weight in each entry.

By the principle of indifference, we have that

the optimal strategy p for I satisfies $\sum_{i=1}^{n} p_i a_{ij} = V$, $j = 1, 2, \cdots, n$. i.e. $p_j \lambda_j = V$, $j = 1, 2, \cdots n$.

Together with the condition $\sum_{j=1}^{n} p_j = 1$, we get $V = \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)^{-1}$.

And $p_j = \frac{V}{\lambda_j}$, i.e. $p_j = \left(\sum_{j=1}^{n} 1/\lambda_j\right)^{-1} \cdot (1/\lambda_j)$, hence $p = \left(\sum_{j=1}^{n} 1/\lambda_j\right)^{-1} (1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_n)$.

Similarly, we can get $q = \left(\sum_{j=1}^{n} 1/\lambda_j\right)^{-1} (1/\lambda_1, 1/\lambda_2, \cdots, 1/\lambda_n)$.

Since $\lambda_n \ge \lambda_{n-1} \ge \cdots \ge \lambda_1 > 0$, then $p = q > 0$, hence p and q are optimal strategies and $v = \left(\sum_{j=1}^{n} 1/\lambda_j\right)^{-1}$ is the value of the game with game matrix D.